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SIMPLICIAL CHIRAL MODELS<sup>★</sup>

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## ABSTRACT

Principal chiral models on a  $d-1$  dimensional simplex are introduced and studied analytically in the large  $N$  limit. The  $d = 0, 2, 4$  and  $\infty$  models are explicitly solved. Relationship with standard lattice models and with few-matrix systems in the double scaling limit are discussed.

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The importance of understanding the large  $N$  limit of matrix-valued field models cannot be overestimated. Not only is this the basic ingredient of the  $1/N$  expansion in the physically relevant case of QCD, but also the existence of large  $N$  criticalities for finite values of the coupling is the starting point for the approach to 2-dimension quantum gravity known as the double scaling limit. Moreover we note that solutions of few-matrix systems may have a direct application to more complex systems in the context of strong-coupling expansion, since they may be reinterpreted as generating functionals for classes of group integrals that are required in strong coupling calculations [1]. Unfortunately our knowledge of exact solutions for the large  $N$  limit of unitary matrix models is still limited. After Gross and Witten's solution [2] of the single-link problem, exact results were obtained only for the external field problem [3,4] and a few toy models ( $L = 3, 4$  chiral chains) [5,6].

In this letter we introduce a new class of lattice chiral models, whose large  $N$  behavior can be analyzed by solving an integral equation for the eigenvalue distribution of a single hermitian semi definite positive matrix. The models we are going to study are principal chiral models, with a global  $U(N) \times U(N)$  symmetry, defined on a  $(d-1)$ -dimensional simplex formed by connecting in a fully symmetric way  $d$  vertices by  $\frac{(d-1)(d-2)}{2}$  links. The partition function for such a system is defined to be

$$Z_d = \int \prod_{i=1}^d dU_i \exp N\beta \sum_{i>j=1}^d \text{Tr} [U_i U_j^\dagger + U_j U_i^\dagger]. \quad (1)$$

Despite its apparent simplicity, this class of models includes most of the previously known solvable systems. As a function of the parameter  $d$ , which specifies the coordination number of lattice sites, it interpolates between the two-dimensional Gross-Witten model (with third order phase transition) and the infinite-dimensional mean field solution (showing a first order phase transition) of standard infinite volume lattice models. It also includes a case which, in the double scaling limit, corresponds to that of a  $c = 1$  conformal field theory.

It is possible to eliminate the direct interactions among the unitary matrices  $U_i$  by introducing an identity in the form

$$1 = \frac{\int dA \exp -N\beta \text{Tr} [A - \sum_{i=1}^d U_i][A^\dagger - \sum_{j=1}^d U_j^\dagger]}{\int dA \exp -N\beta \text{Tr} AA^\dagger}, \quad (2)$$

where  $A$  is a  $N \times N$  complex matrix. As a consequence we obtain  $Z_d = \tilde{Z}_d/\tilde{Z}_0$  where

$$\tilde{Z}_d = \int \prod_{i=1}^d dU_i dA \exp \left\{ -N\beta \text{Tr} AA^\dagger + N\beta \text{Tr} A \sum_i U_i^\dagger + N\beta \text{Tr} A^\dagger \sum_i U_i - N^2\beta d \right\}. \quad (3)$$

We now introduce the function

$$F[BB^\dagger] = \frac{1}{N^2} \log \int dU \exp \frac{N}{2} \text{Tr} [BU^\dagger + UB^\dagger]. \quad (4)$$

$F$  is a known function of the eigenvalues  $x_i$  of the hermitian semipositive definite matrix  $BB^\dagger$ . More specifically, in the large  $N$  limit, we know that [3,4]

$$F(x_i) = \frac{1}{N} \sum_i (r + x_i)^{1/2} - \frac{1}{2N^2} \sum_{i,j} \log \left[ \frac{(r + x_i)^{1/2} + (r + x_j)^{1/2}}{2} \right] - \frac{r}{4} - \frac{3}{4}, \quad (5)$$

and there are two distinct phases:

- a) weak coupling  $r = 0$ ,
- b) strong coupling  $\frac{1}{N} \sum_i (r + x_i)^{-1/2} = 1$ .

Up to irrelevant factors, it follows that

$$\tilde{Z}_d = \int dB \exp \left\{ -\frac{N}{4\beta} \text{Tr} BB^\dagger + N^2 d F(BB^\dagger) - N^2\beta d \right\}, \quad (6)$$

where  $B$  replaces  $2\beta A$ .

Morris [7] has shown that the angular integration can be performed in the case of complex matrices, and, again up to irrelevant factors, we may replace eq. (6) by

$$\tilde{Z}_d = \int_0^\infty dx_i \prod_{i \neq j} (x_i - x_j) \exp \left\{ -\frac{N}{4\beta} \sum_i x_i + N^2 d F(x_i) - N^2 \beta \right\}. \quad (7)$$

In the large  $N$  limit it is legitimate to evaluate this integral by a saddle point method. The saddle-point equation resulting from eqs. (5) and (7) is

$$\frac{\sqrt{r+x_i}}{2\beta} - d = \frac{1}{N} \sum_{j \neq i} \frac{(4-d)\sqrt{r+x_i} + d\sqrt{r+x_j}}{x_i - x_j}. \quad (8)$$

We introduce a new variable  $z_i \equiv \sqrt{r+x_i}$  and, in the large  $N$  limit, we assume that these eigenvalues lie in a single interval  $[a, b]$ . Denoting the large  $N$  eigenvalue density by  $\rho(z)$ , eq. (8) becomes an integral equation for  $\rho(z)$ ,

$$\frac{z}{2\beta} - d = \int_a^b dz' \rho(z') \left[ \frac{2}{z - z'} - \frac{(d-2)}{z + z'} \right], \quad (9)$$

where the integration region is restricted by the condition,  $0 \leq a \leq b$ , with  $a$  and  $b$  determined dynamically. In particular, the normalization condition,

$$\int_a^b \rho(z') dz' = 1, \quad (10)$$

must always be satisfied. Furthermore, one has the following constraint,

$$\int_a^b \rho(z') \frac{dz'}{z'} \leq 1, \quad (11)$$

with the equality holding exactly in the strong coupling region where  $a = \sqrt{r}$ .

Let us begin by first discussing several simple cases where eq. (9) can be solved readily. For  $d = 0$ , the problem reduces to one with a pure gaussian interaction, and, by a more or less standard technique, one finds that

$$\rho(z) = \frac{1}{4\pi\beta} \sqrt{16\beta - z^2}. \quad (12)$$

For  $d = 0$ , there is no weak coupling phase. Note also that, since the  $F$ -term in eq. (6) vanishes for  $d = 0$ , eq. (12) is obtained with  $a = 0$ . As a consequence, up to a constant,

$$\tilde{Z}_0 = \exp N^2 \ln \beta, \quad (13)$$

as expected.

When  $d = 2$  we obtain

$$\begin{aligned} \rho_w(z) &= \frac{1}{4\pi\beta} \sqrt{8\beta - (z - 4\beta)^2} & \beta &\geq \frac{1}{2}, \\ \rho_s(z) &= \frac{1}{4\pi\beta} z \sqrt{\frac{(1 + 6\beta) - z}{z - (1 - 2\beta)}}, & r(\beta) &= (1 - 2\beta)^2, & \beta &\leq \frac{1}{2}, \end{aligned} \quad (14)$$

and one may show that all results are consistent with a reinterpretation of the model as a Gross-Witten [2] one-plaquette model, with  $\beta_c = \frac{1}{2}$ .

When  $d = 3$  the model can be mapped into the three-link chiral chain, which is known to possess a third order phase transition at  $\beta_c = \frac{1}{3}$  [5].

The first non-trivial and new situation begins at  $d = 4$ . We have explicitly solved the  $d = 4$  model, both in the weak and in the strong coupling phase. The eigenvalue density may be expressed in terms of elliptic integrals, and supplementary conditions allow for the determination of  $a$  and  $b$ .

Introducing the variable  $k(\beta) = \sqrt{1 - \frac{a^2}{b^2}}$ , in weak coupling we obtain, in terms

of standard elliptic integrals  $K$ ,  $\Pi$ , and  $E$ ,

$$\rho_w(z) = \frac{b}{2\pi^2\beta} (z^2 - a^2)^{1/2} (b^2 - z^2)^{-1/2} \left[ K(k) - \frac{z^2}{b^2} \Pi\left(1 - \frac{z^2}{b^2}, k\right) \right], \quad (15)$$

with the condition  $4\pi\beta = bE(k)$ . In strong coupling we have

$$\rho_s(z) = \frac{1}{2\pi^2\beta} \frac{z^2}{b} (z^2 - a^2)^{-1/2} (b^2 - z^2)^{-1/2} \left[ (b^2 - a^2)K(k) - (z^2 - a^2)\Pi\left(1 - \frac{z^2}{b^2}, k\right) \right], \quad (16)$$

with the condition  $4\pi\beta = b[E(k) - \frac{a^2}{b^2}K(k)]$ . In both regimes, eq. (10) must also be satisfied. Closed form solutions for the constraints may be obtained at criticality: when  $\beta = \beta_c = \frac{1}{4}$ , we get  $a = 0$ ,  $b = \pi$ , and

$$\rho_c(z) = \frac{z}{\pi^2} \log \frac{1 + \sqrt{1 - \frac{z^2}{\pi^2}}}{1 - \sqrt{1 - \frac{z^2}{\pi^2}}}. \quad (17)$$

Let's finally observe that a large- $d$  solution of eq. (9) may easily be found by assuming  $\rho(z) \rightarrow \delta(z - \bar{z})$ . The weak coupling solution is

$$\bar{z} = \beta d \left[ 1 + \sqrt{1 - \frac{1}{\beta d}} \right], \quad \beta \geq \beta_c = \frac{1}{d}, \quad (18)$$

and for strong coupling  $\bar{z} = 0$ . Amusingly enough, this solution turns out to coincide with the large  $D \equiv \frac{d}{2}$  mean field solution [8] of infinite volume principal chiral models on D-dimensional hypercubic lattices with the same coordination number as our corresponding models.

We would like to add a few comments. Solving eq. (9) is certainly a well defined problem for any value of  $d$ , and in particular we expect to be able to find explicit solutions for simple cases, like  $d = 1$  and  $d = 3$ . It is also possible to analyze eq. (9) numerically; details of our analytical and numerical techniques will be reported elsewhere; we only mention that for sufficiently large  $\beta > \beta_c$  we can get

the eigenvalue distribution with desired accuracy, while near criticality convergence is slow: however within 1% accuracy we have evidence that  $\beta_c = \frac{1}{d}$  for all integer values of  $d$  [9]. It would be quite interesting to achieve more information, both qualitative and quantitative, on the  $d$ -dependence of the phase transition.

The thermodynamical quantity whose computation is easiest is the internal energy per unit link,  $w_1$ , which may be obtained from

$$d(d-1)w_1 = \frac{1}{4\beta^2} \int_a^b dz \rho(z)(z^2 - r) - d - \frac{1}{\beta}. \quad (19)$$

One may then extract, in the vicinity of  $\beta_c$ , the critical exponent for the specific heat,  $\alpha$ . At present we know that when  $d = 2$ ,  $\alpha = -1$ , when  $d = 3$ ,  $\alpha = -\frac{1}{2}$ , when  $d = 4$ ,  $\alpha = 0$ , and for sufficiently large  $d$  the transition is first-order, *i.e.*,  $\alpha = 1$ .

It is worth observing in this context that a more general model involving four unitary matrices and three couplings, interpolating between our  $d = 4$  case and the 4-link chiral chain, can be re-expressed as a model of two coupled complex matrices and admits many solvable limits, all characterized by  $\alpha = 0$ , which corresponds to a  $c = 1$  conformal field theory.

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